MTH 304: Metric Spaces and Topology

Homework III

(Due 06/02)

1. For $n \geq 1$, let
$$P_n : S^n \setminus \{x\} \to \mathbb{R}^n$$
be the stereographic projection.
   (a) Derive an explicit expression for P_n, assuming that x is the north pole.
   (b) Show that P_n is a homeomorphism, and hence $S^n \setminus \{x\} \approx \mathbb{R}^n$.

2. From real analysis, you know that $f : \mathbb{R} \to \mathbb{R}$ is continuous if for given $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$, whenever $|x - a| < \delta$. Show that this is equivalent to the topological formulation of continuity in this setting.

3. Let $B(x, r)$ be an open ball in $\mathbb{R}^n$ with center x and radius $r > 0$. Show that
$$B(x, r) \approx \mathbb{R}^n.$$

4. Let X be a topological space, and $f, g : X \to \mathbb{R}$ be continuous maps.
   (a) Show that $\{x \mid f(x) \leq g(x)\}$ is closed in X.
   (b) Show that $h = \min(f, g)$ is continuous. [Hint: Use the pasting lemma.]

5. Will the pasting lemma (see Lesson plan 1.7 (xiv)) hold true if we assume that A and B are open sets? Why or why not?

6. An indexed family $\{A_\alpha\}_{\alpha \in J}$ of subsets of topological space $(X, \mathcal{T}_X)$ is said to be *locally finite* if each $x \in X$ has a neighborhood U such that
$$|\{\alpha \in J \mid U \cap A_\alpha \neq \emptyset\}| < \infty.$$
Let $\{A_\alpha\}_{\alpha \in J}$ be a locally finite family of subsets in X such that $X = \cup_{\alpha \in J} A_\alpha$. If $f : X \to Y$ is a map such that for each α, A_α is closed and $f|_{A_\alpha}$ is continuous, then show that f is continuous.

7. Let $A \subset X$, Y a Hausdorff space, and $f : A \to Y$ be continuous. If f can be extended to a continuous function $g : \bar{A} \to Y$, then show that g is uniquely determined by f.